Lightface Π^0_3-completeness of density sets under effective Wadge reducibility

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Abstract. Let $\mathcal{A} \subseteq {}^\omega 2$ be measurable. The density set $D\mathcal{A}$ is the set of $Z \in {}^\omega 2$ such that the local measure of $\mathcal{A}$ along Z tends to 1. Suppose that $\mathcal{A}$ is a Π^0_1 set with empty interior and the uniform measure of $\mathcal{A}$ is a positive computable real. We show that $D\mathcal{A}$ is lightface Π^0_3 complete for effective Wadge reductions. This is an algorithmic version of a result in descriptive set theory by Andretta and Camerlo [1]. They show a completeness result for boldface Π^0_3 sets under plain Wadge reductions.

1 Introduction

We work in Cantor space ${}^\omega 2$ with the product measure μ. For a finite bit string s and a measurable set $\mathcal{A} \subseteq {}^\omega 2$, the local measure of $\mathcal{A}$ above s is

$$\mu([s] \cap \mathcal{A})/2^{-|s|},$$

where $[s]$ is the clopen set in ${}^\omega 2$ consisting of all the extensions of s (this set is often denoted N_s in the literature). Let $D\mathcal{A}$ be the points Z of such that $\mathcal{A}$ has density 1 at Z, namely

$$D\mathcal{A} = \{Z \colon \lim_n \mu_{Z \upharpoonright n}(\mathcal{A}) = 1\}.$$

We call *density set* every set of the form $D\mathcal{A}$, for some measurable A. We will be mainly interested in closed $\mathcal{A}$, in which case $D\mathcal{A} \subseteq \mathcal{A}$.

The Lebesgue density theorem for Cantor space states that if $\mathcal{A}$ is measurable, then almost every $Z \in \mathcal{A}$ is in $D\mathcal{A}$. This result has been the seed for recent investigations both in algorithmic randomness and in descriptive set theory.

In algorithmic randomness, density has been instrumental in solving the long-open "covering problem". Combining Bienvenu et al. [3] and Day, Miller [5] yielded the answer; for an overview see [2]. Khan [6] obtained a variety of results, in particular relating density for the reals with density for Cantor space. In a recent article, Myabe et al. [7] define density randomness of a point Z in Cantor space as the combination of Martin-Löf-randomness of Z and the property that $Z \in D\mathcal{P}$ for each Π^0_1 set $\mathcal{P}$ containing Z; they show the equivalence of density randomness with a number of notions stemming from effective analysis. Martin-Löf-randomness of Z does not necessarily imply that Z satisfies the effective version of Lebesgue's theorem; for instance, the least element of a non-empty Π^0_1 set $\mathcal{P}$ of ML-randoms is not in $D\mathcal{P}$.

In descriptive set theory, Andretta and Camerlo [1] have analised the complexity of the density sets. It is easily seen that the $\mathbf{\Pi}^0_3$ pointclass is an upper bound for this study. For example, if $A = [s]$, then $D\mathcal{A} = A$, which is closed. In [1, sec. 7] the authors show that $D\mathcal{A}$ is $\mathbf{\Pi}^0_3$-complete with respect to Wadge reducibility in case that $\mathcal{A}$ has empty interior and positive measure. They also prove that density sets can have any complexity within $\mathbf{\Pi}^0_3$. The first author in [4] has conducted a similar study of the difference hierarchy over the closed sets in the setting of the real line with the Lebesgue measure.

The goal of this short paper is to connect the two approaches to Lebesgue's theorem. We give an algorithmic version of the result in [1]. We show that for any (lightface) Π^0_1 set $\mathcal{A} \subseteq {}^\omega 2$ with empty interior and measure $\mu\mathcal{A}$ a positive computable real, the density set $D\mathcal{A}$ is lightface Π^0_3 complete for algorithmic Wadge reductions.

To say that a real r is computable means that from a number $n \in \omega$ we can compute a rational that is within 2^{-n} of r. The algorithmic version of Wadge reducibility is as follows: for $\mathcal{C}, \mathcal{D} \subseteq {}^\omega 2$, we write $\mathcal{D} \leq_m \mathcal{C}$ if there is a total Turing functional Ψ such that $\mathcal{D} = \Psi^{-1}(\mathcal{C})$. Totality means that $\Psi(Y) \in {}^\omega 2$ for each Y; equivalently, $\Psi(Y; n)$ is obtained by evaluating a truth table computed from n on Y. Note that it is easy to construct such a set $\mathcal{A}$, for instance by a Cantor space version of the construction of a Cantor set of positive measure in the unit interval.

In our proof, while we import the basic combinatorics of the approach in [1], the details are more complicated because we need to build an algorithmic Wadge reduction. This is where we use the hypothesis that the measure of $\mathcal{A}$ is computable. It is not known at present whether this hypothesis is actually necessary. The other hypothesis (that the interior be empty) is of course necessary, as for instance shown by taking $\mathcal{A}$ to be the whole Cantor space.

2 Completeness for lightface point classes

For the basics about arithmetical hierarchy see for example [10] and [9].

Definition 1. *Consider a lightface point class Γ in Cantor space, such as Π^0_2 or Π^0_3. We say that $\mathcal{C} \subseteq {}^{\omega}2$ is Γ-complete if $\mathcal{C}$ is in Γ, and for each $\mathcal{D} \in \Gamma$ we have $\mathcal{D} \leq_m \mathcal{C}$.*

The next result is folklore (see e.g. [8]). For the reader's benefit we provide the short proof.

Proposition 2 *The class $\mathcal{C}$ of all sequences with infinitely many 1's is Π^0_2-complete.*

Proof. The class $\mathcal{C}$ is clearly Π^0_2. Now suppose $\mathcal{D}$ is Π^0_2, so $\mathcal{D} = \bigcap_n \mathcal{G}_n$ with $\mathcal{G}_{n+1} \subseteq \mathcal{G}_n$ for a uniformly Σ^0_1 sequence $\langle \mathcal{G}_n \rangle_{n\in\omega}$. Define a total Turing functional Ψ as follows. Given a sequence of bits Z, at stage t we use the first t bits of Z, and append one bit at the end of the output Ψ^Z. We append 0 unless we see at stage t that $[Z\upharpoonright_t] \subseteq \mathcal{G}_n$ for the next n; in that case we append 1.

In the following we effectively identify ${}^{\omega\times\omega}2$ and ${}^{\omega}2$ via the standard computable pairing function $\omega \times \omega \to \omega$. The following is presumably folklore.

Proposition 3 *The set $\mathcal{E} = \{Z \in {}^{\omega}2\colon \forall n \forall^{\infty} k\, Z(n,k) = 0\}$ is Π^0_3-complete.*

Proof. Clearly $\mathcal{E}$ is Π^0_3. Now suppose a given set $\mathcal{F}$ is Π^0_3, so $\mathcal{G} = \bigcap_n \mathcal{G}_n$ where $\mathcal{G}_n$ is uniformly Σ^0_2. By the previous proposition and the uniformity in $\mathcal{D}$ of its completeness part, for each n we effectively have a Turing functional Ψ_n such that $\mathcal{G}_n = \Psi_n^{-1}({}^{\omega}2\backslash\mathcal{C})$. Define a total Turing functional $\Psi\colon {}^{\omega}2 \to {}^{\omega\times\omega}2$ by

$$\Psi^Z(n,r) = \Psi^Z_n(r).$$

Clearly $Z \in \mathcal{F} \Leftrightarrow \forall n\, [Z \in \mathcal{G}_n] \Leftrightarrow \forall n\, [\Psi^Z_n \in {}^{\omega}2 \setminus \mathcal{C}] \Leftrightarrow \Psi^Z \in \mathcal{E}$.

3 Reaching the maximal complexity

Theorem 4 *Let $\mathcal{A} \subseteq {}^{\omega}2$ be a Π^0_1 set with empty interior such that $\mu\mathcal{A} > 0$ and $\mu\mathcal{A}$ is a computable real. Then $D\mathcal{A}$ is Π^0_3-complete.*

We begin with some preliminaries. In the following s, t, u denote strings of bits. Note that if $\mu\mathcal{A}$ is a computable real then $\mu_s(\mathcal{A})$ is a computable real uniformly in s (see e.g. [9, 1.9.18]). Also, $\mu_s(\mathcal{A}) < 1$ for each s because $\mathcal{A}$ has empty interior.

Let $L(s) = \mu_s(\mathcal{A})$. We note that L is a computable martingale in the sense of algorithmic randomness. That is, $L \geq 0$, the "martingale equality" $L(s0) + L(s1) = 2L(s)$ holds for each string s, and $L(s)$ is a computable real uniformly in s; see e.g. [9, Ch. 7]. By hypothesis that $\mathcal{A}$ has empty interior, we have $L(s) < 1$ for each s.

We will show that the oscillation behaviour of L along certain paths can be controlled sufficiently well in order to code the Π^0_3-complete set $\mathcal{E}$ from Proposition 3 into $D\mathcal{A}$. For $p \in \mathbb{N}$ let $\delta_p = 1 - 3^{-p}$. We write $\theta(s) = p$ if $\delta_{p-1} < L(s) < \delta_p$. We leave $\theta(s)$ undefined in case $L(s) = \delta_k$ for some k. In the following, when we write $\theta(s)$ we imply that $\theta(s)$ is defined. Observe that, if $\theta(X \upharpoonright_n)$ is defined for each n and $\lim_n \theta(X \upharpoonright n) = \infty$, then $X \in D\mathcal{A}$. Note that the binary relations $\{\langle s, p\rangle : \theta(s) = p\}$ and $\{\langle s, p\rangle : \theta(s) \geq p\}$ are Σ^0_1 because the martingale L is computable.

Lemma 5.

(i) (Increasing the value of L.) *Let $p < k$ and $\theta(s) = p$. There is $t \supset s$ such that $\theta(t) \geq k$ and $L(u) > \delta_{p-1}$ for each u with $s \subseteq u \subseteq t$.*
(ii) (Decreasing the value of L.) *Let $p > q$ and $\theta(s) = p$. There is $t \supset s$ such that $\theta(t) = q$ and $L(u) > \delta_{q-1}$ for each u with $s \subseteq u \subseteq t$.*

Proof. (i) By an application of the Lebesgue density theorem [1, Prop. 3.5] (where r there is δ_k), there exists a prefix minimal string $v \supset s$ such that $L(v) \geq \delta_k$ and $L(u) \geq L(s)$ for each u with $s \subseteq u \subseteq v$. If $\theta(v)$ is defined, the string $t = v$ is as required. Otherwise we have $L(v) = \delta_m$ for some $m \geq k$. By the Lebesgue density theorem, L is not constant on the set of extensions of v, so one can choose a prefix minimal string $w \supseteq v$ such that $L(w0) \neq L(w1)$. By the minimality of w we have $L(v') = \delta_m$ for each v' with $v \subseteq v' \subseteq w$.

First suppose that $L(w0) > L(w)$. Since $L(w0) < 1$, we have $L(w0) - L(w) < 3^{-m}$. Hence $L(w) - L(w1) = L(w0) - L(w) < 3^{-m}$ by the martingale equality, and thus $\delta_{m-1} < L(w1) < \delta_m$ because $\delta_{m-1} = \delta_m - 2 \cdot 3^{-m}$. Then $\theta(w1) = m \geq k$, so the string $t = w1$ is as required. If $L(w1) > L(w)$ instead, then $t = w0$ is as required by an analogous argument.
(ii) As $L(s) < 1$, by the Lebesgue density theorem there exists a prefix minimal $t \supset s$ such that $L(t) < \delta_q$. Let $t = vb$ where $b \in \{0, 1\}$. Since $L \leq 1$ the martingale equality for $1 - L$ implies that $2(1 - L(v)) \geq 1 - L(t)$. So $L(t) \leq \delta_{q-1}$ would imply $2(1 - L(v)) \geq 1 - L(t) \geq 3 \cdot 3^{-q} > 2 \cdot 3^{-q}$, and hence $L(v) < \delta_q$ contrary to the minimality of t. Hence $\theta(t) = q$ and $L(u) > \delta_{q-1}$ for each u with $s \subseteq u \subseteq t$, as required. This establishes the lemma.

Remark 6. By the remarks on θ above and since L is computable, all the conditions in the Lemma are Σ^0_1. Thus in (i) given a string s and numbers $p < k$, if $\theta(s) = p$ we can run a search for the string t. So the function $s, p, k \mapsto t$ is partial computable and defined whenever $\theta(s) = p < k$. A similar remark applies to (ii).

Proof (of Theorem 4). By Proposition 3, the set

$$\mathcal{E} = \{Z \in {}^{\omega\times\omega}2 \mid \forall q \forall^\infty n[Z(q, n) = 0]\}$$

is Π^0_3-complete; thus, it will be enough to show that $\mathcal{E} \leq_m D(\mathcal{A})$. In the following let a, b denote bit-valued square matrices. Given such a matrix a, we will compute a string $s = \psi(a) \in {}^{<\omega}2$ in such a way that $\theta(s)$ is defined. We ensure that $a \subseteq b \to \psi(a) \subseteq \psi(b)$. Then the function

$$\Psi : {}^{\omega\times\omega}2 \to {}^{\omega}2, \qquad \Psi(Z) = \bigcup_n \psi(Z \upharpoonright n \times n)$$

is a total Turing functional.
Defining ψ. The definition of ψ is by recursion, using Lemma 5 and Remark 6. If $\mu\mathcal{A} > 1/2$, after removing from $\mathcal{A}$ either the elements of Cantor space starting with 0 or the elements starting with 1, we may assume that $\mu\mathcal{A} \leq 1/2$. So we may assume that $\theta(\emptyset)$ is defined. We set $\psi(\emptyset) = \emptyset$.

Now suppose that $\phi(a)$ has been defined for each $n \times n$ bit-valued matrix a in accordance with the conditions above. Given a matrix $b = \langle b(i, j) \mid i, j < n + 1\rangle$, let $s = \psi(a)$ where $a = b \upharpoonright n \times n$. Then $p = \theta(s)$ is defined.

If there is a $q \leq n$ such that $b(q, n) = 1$, choose q least.

- If $q < p$, via (ii) of Lemma 5 compute a string $t \supset s$ such that $\theta(t) = q$ and $L(u) > \delta_{q-1}$ for each u with $s \subseteq u \subseteq t$, and define $\psi(b) = t$.
- If $q \geq p$, or there is no such $q \leq n$ at all, let $k = \max(p + 1, n + 1)$. Via (i) of Lemma 5 compute a string $t \supset s$ such that $\theta(t) \geq k$ and $L(u) > \delta_{p-1}$ for each u with $s \subseteq u \subseteq t$, and define $\psi(b) = t$. This completes the recursion step.

We verify that $\mathcal{E} \leq_m D\mathcal{A}$ via Ψ. First suppose that $Z \notin \mathcal{E}$. Let q be least such that $\exists^\infty n[Z(q, n) = 1]$. Then for arbitrarily large n, we have $\theta(\psi(Z \upharpoonright n \times n)) = q$, hence $\theta(\Psi(Z) \upharpoonright r) = q$ for infinitely many r. Therefore $\Psi(Z) \notin D\mathcal{A}$.

Now suppose that $Z \in \mathcal{E}$. Then for every $q \in \omega$, there is m_q such that $\forall m \geq m_q[Z(q, m) = 0]$. For $r \in \omega$, let $n_r = \max\{m_0, \ldots, m_r, r\}$. Note that for each $n \geq n_r$, if $Z(q, n) = 1$ then $q > r$.

The following claim will show that $\Psi(Z) \in D\mathcal{A}$.

Claim. Given $n > n_r$, let $s = \psi(Z \upharpoonright n \times n)$ and $t = \psi(Z \upharpoonright (n+1) \times (n+1))$. For each string u with $s \subseteq u \subseteq t$, we have $L(u) \geq \delta_r$.

To see this, we prove inductively that $\theta(\psi(Z \upharpoonright n \times n)) > r$ for each $n > n_r$. For the start of the induction at $n_r + 1$, suppose that $s = \psi(Z \upharpoonright n_r \times n_r)$ has just been defined and consider the next step of the definition of ψ along Z. The least possible value of q is $r + 1$. Since $n_r \geq r$, no matter whether we apply (ii) or (i) of the lemma we ensure that $\theta(t) > r$ and hence $L(t) > \delta_r$, where $t = \psi(Z \upharpoonright (n_r + 1) \times (n_r + 1))$.

For the inductive step, suppose that $n > n_r$ and for $s = \psi(Z \upharpoonright n \times n)$ we have $\theta(s) > r$. Let $t = \psi(Z \upharpoonright (n + 1) \times (n + 1))$. Again the least possible value of q is $r + 1$. If we apply (ii) of the lemma then $\theta(t) > r$ and $L(u) > \delta_r$ for each u with $s \subseteq u \subseteq t$. If we apply (i) then, where $\theta(s) = p > r$, we have $\theta(t) \geq p + 1$ and $L(u) > \delta_r$ for each u with $s \subseteq u \subseteq t$. This completes the claim.

We conclude that for each r, we have $L(\Psi(Z) \upharpoonright_m) \geq \delta_r$ for sufficiently large m. Hence $\Psi(Z) \in D\mathcal{A}$.

Theorem 4 leaves open the following question. Is there a Π^0_1 class with empty interior and non-computable measure such that the density set is not Π^0_3-complete?

References

1. A. Andretta and R. Camerlo. The descriptive set theory of the Lebesgue density theorem. *Advances in Mathematics*, 234:1–42, 2013.
2. L. Bienvenu, A. Day, N. Greenberg, A. Kučera, J. Miller, A. Nies, and D. Turetsky. Computing K-trivial sets by incomplete random sets. *Bull. Symb. Logic*, 20:80–90, 2014.
3. L. Bienvenu, N. Greenberg, A. Kučera, A. Nies, and D. Turetsky. Coherent randomness tests and computing the K-trivial sets. *Journal European Mathematical Society*, 18:773–812, 2016.
4. C. Carotenuto. On the topological complexity of the density sets of the real line. PhD thesis, Università di Salerno, 2015. http://elea.unisa.it:8080/jspui/bitstream/10556/1972/1/tesi_G.Carotenuto.pdf
5. A. R. Day and J. S. Miller. Density, forcing and the covering problem. *Mathematical Research Letters*, 22(3):719–727, 2015.
6. M. Khan. Lebesgue density and Π^0_1-classes. J. Symb. Logic, to appear.
7. K. Miyabe, A. Nies, and J. Zhang. Using almost-everywhere theorems from analysis to study randomness. Bulletin Symb. Logic, to appear, 2016.
8. F. Montagna and A. Sorbi. Creativeness and Completeness in Recursion Categories of Partial Recursive Operators. *J. Symbolic Logic*, 54: 1023–1041, 1989.
9. A. Nies. *Computability and randomness*, volume 51 of *Oxford Logic Guides*. Oxford University Press, Oxford, 2009. 444 pages. Paperback version 2011.
10. P. Odifreddi. *Classical Recursion Theory (Volume I)*. NorthHolland Publishing Co., Amsterdam, 1989.